

Locally compact groups and continuous logic

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Abstract We study expressive power of continuous logic in classes of (locally compact) groups. We also describe locally compact groups which are separably categorical structures.

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1 Introduction

In this paper we give very concrete applications of continuous logic in group theory. We consider classes of (locally compact) metric groups which can be also viewed as (reducts of) axiomatizable classes of continuous structures. Then by some standard logical tricks we obtain several interesting consequences. Usually we concentrate on classes which are typical in geometric group theory.

The following notion is one of the main objects of the paper. A class of groups \mathcal{K} is called *bountiful* if for any pair of infinite groups $G \leq H$ with $H \in \mathcal{K}$ there is $K \in \mathcal{K}$ such that $G \leq K \leq H$ and $|G| = |K|$. It was introduced by Ph.Hall and was studied in papers [15], [17], [20] and [23]. Some easy logical observations from [15] show that if \mathcal{K} is a reduct of a class axiomatizable in $L_{\omega_1\omega}$ then \mathcal{K} is bountiful.

When one considers topological groups, the definition of bountiful classes should be modified as follows.

Definition 1.1 *A class of topological groups \mathcal{K} is called bountiful if for any pair of infinite groups $G \leq H$ with $H \in \mathcal{K}$ there is $K \in \mathcal{K}$ such that $G \leq$*

$K \leq H$ and the density character of G (i.e. the smallest cardinality of a dense subset of the space) coincides with the density character of K .

We mention paper [11] where similar questions were studied in the case of locally compact groups. We will see below that under additional assumptions of metrizability logical tools become helpful in this class of groups. We should only replace first-order logic (or $L_{\omega_1\omega}$) by continuous one. We concentrate on negations of properties **(T)**, **FH**, **FR** ([1], [10]) and on negations of boundedness properties classified in [19].

In the final part of the paper we consider separable locally compact groups which have separably categorical continuous theory, i.e. the group is determined uniquely (up to metric isomorphism) by its continuous theory and the density character. It is interesting that some basic properties of the automorphism groups of such structures are strongly connected with some classes examined on bountifulness below.

In the rest of this introduction we briefly remind the reader some preliminaries of continuous logic. Then we finish this section by some remarks on sofic groups.

Continuous structures. We fix a countable continuous signature

$$L = \{d, R_1, \dots, R_k, \dots, F_1, \dots, F_l, \dots\}.$$

Let us recall that a *metric L -structure* is a complete metric space (M, d) with d bounded by 1, along with a family of uniformly continuous operations on M and a family of predicates R_i , i.e. uniformly continuous maps from appropriate M^{k_i} to $[0, 1]$. It is usually assumed that to a predicate symbol R_i a continuity modulus γ_i is assigned so that when $d(x_j, x'_j) < \gamma_i(\varepsilon)$ with $1 \leq j \leq k_i$ the corresponding predicate of M satisfies

$$|R_i(x_1, \dots, x_j, \dots, x_{k_i}) - R_i(x_1, \dots, x'_j, \dots, x_{k_i})| < \varepsilon.$$

It happens very often that γ_i coincides with id . In this case we do not mention the appropriate modulus. We also fix continuity moduli for functional symbols. Note that each countable structure can be considered as a complete metric structure with the discrete $\{0, 1\}$ -metric.

By completeness continuous substructures of a continuous structure are always closed subsets.

Atomic formulas are the expressions of the form $R_i(t_1, \dots, t_r)$, $d(t_1, t_2)$, where t_i are terms (built from functional L -symbols). In metric structures they can take any value from $[0, 1]$. *Statements* concerning metric structures are usually formulated in the form

$$\phi = 0$$

(called an *L-condition*), where ϕ is a *formula*, i.e. an expression built from 0,1 and atomic formulas by applications of the following functions:

$$x/2, x \dot{-} y = \max(x - y, 0), \min(x, y), \max(x, y), |x - y|, \\ \neg(x) = 1 - x, x \dot{+} y = \min(x + y, 1), \sup_x \text{ and } \inf_x.$$

A *theory* is a set of *L-conditions* without free variables (here \sup_x and \inf_x play the role of quantifiers).

It is worth noting that any formula is a γ -uniformly continuous function from the appropriate power of M to $[0, 1]$, where γ is the minimum of continuity moduli of *L*-symbols appearing in the formula.

The following theorem is one of the main tools of this paper.

Löwenheim-Skolem Theorem. ([3], Proposition 7.3) *Let κ be an infinite cardinal number and assume $|L| \leq \kappa$. Let M be an *L*-structure and suppose $A \subset M$ has density $\leq \kappa$. Then there exists a substructure $N \subseteq M$ containing A such that $\text{density}(N) \leq \kappa$ and N is an elementary substructure of M , i.e. for every *L*-formula $\phi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in N$ the values of $\phi(a_1, \dots, a_n)$ in N and in M are the same.*

Remark 1.2 It is proved in [11] that for any locally compact group G , the entire interval of cardinalities between \aleph_0 and $w(G)$, the weight of the group, is occupied by the weights of closed subgroups of G . We remind the reader that the weight of a topological space (X, τ) is the smallest cardinality which can be realized as the cardinality of a basis of (X, τ) . If the group G is metric, the weight of G coincides with the density character of G . This yields the following version of the Löwenheim-Skolem Theorem.

Let G be a metric locally compact group. Then for any cardinality $\kappa < \text{density}(G)$ there is a closed subgroup $H < G$ such that $\text{density}(H) = \kappa$ and H is an elementary substructure of G .

Definability in continuous structures is introduced as follows.

Definition 1.3 *Let $A \subseteq M$. A predicate $P : M^n \rightarrow [0, 1]$ is definable in M over A if there is a sequence $(\phi_k(x) : k \geq 1)$ of *L*(*A*)-formulas such that predicates interpreting $\phi_k(x)$ in M converge to $P(x)$ uniformly in M^n .*

We define the automorphism group $\text{Aut}(M)$ of M to be the subgroup of $\text{Iso}(M, d)$ consisting of all isometries preserving the values of atomic formulas. It is easy to see that $\text{Aut}(M)$ is a closed subgroup with respect to the pointwise convergence topology on $\text{Iso}(M, d)$.

The following statement is Corollary 9.11 of [3].

Let M be an *L*-structure with $A \subseteq M$ and suppose $P : M^n \rightarrow [0, 1]$ is a predicate. Then P is definable in M over A if and

only if whenever (N, Q) is an elementary extension of (M, P) , the predicate Q is invariant under all automorphisms of N that leave A fixed pointwise.

A tuple \bar{a} from M^n is *algebraic* in M over A if there is a compact subset $C \subseteq M^n$ such that $\bar{a} \in C$ and the distance predicate $\text{dist}(\bar{x}, C)$ is definable in M over A . Let $\text{acl}(A)$ be the set of all elements algebraic over A . In continuous logic the concept of algebraicity is parallel to that in traditional model theory (see Section 10 of [3]).

For every $c_1, \dots, c_n \in M$ and $A \subseteq M$ we define the n -type $\text{tp}(\bar{c}/A)$ of \bar{c} over A as the set of all \bar{x} -conditions with parameters from A which are satisfied by \bar{c} in M . Let $S_n(T_A)$ be the set of all n -types over A of the expansion of the theory T by constants from A . There are two natural topologies on this set. The *logic topology* is defined by the basis consisting of sets of types of the form $[\phi(\bar{x}) < \varepsilon]$, i.e. types containing some $\phi(\bar{x}) \leq \varepsilon'$ with $\varepsilon' < \varepsilon$. The logic topology is compact.

The d -topology is defined by the metric

$$d(p, q) = \inf\{\max_{i \leq n} d(c_i, b_i) \mid \text{there is a model } M \text{ with } M \models p(\bar{c}) \wedge q(\bar{b})\}.$$

By Propositions 8.7 and 8.8 of [3] the d -topology is finer than the logic topology and $(S_n(T_A), d)$ is a complete space.

Separable categoricity. A theory T is *separably categorical* if any two separable models of T are isomorphic. By Theorem 12.10 of [3] a complete theory T is separably categorical if and only if for each $n > 0$, every n -type p is principal. The latter means that for every model $M \models T$, the predicate $\text{dist}(\bar{x}, p(M))$ is definable over \emptyset .

Another property equivalent to separable categoricity states that for each $n > 0$, the metric space $(S_n(T), d)$ is compact. In particular for every n and every ε there is a finite family of principal n -types p_1, \dots, p_m so that their ε -neighbourhoods cover $S_n(T)$.

In first order logic a countable structure M is ω -categorical if and only if $\text{Aut}(M)$ is an *oligomorphic* permutation group, i.e. for every n , $\text{Aut}(M)$ has finitely many orbits on M^n . In continuous logic we have the following modification.

Definition 1.4 *An isometric action of a group G on a metric space (\mathbf{X}, d) is said to be approximately oligomorphic if for every $n \geq 1$ and $\varepsilon > 0$ there is a finite set $F \subset \mathbf{X}^n$ such that*

$$G \cdot F = \{g\bar{x} : g \in G \text{ and } \bar{x} \in F\}$$

is ε -dense in (\mathbf{X}^n, d) .

Assuming that G is the automorphism group of a non-compact separable continuous metric structure M , G is approximately oligomorphic if and only if the structure M is separably categorical (C. Ward Henson, see Theorem 4.25 in [21]). It is also known that separably categorical structures are *approximately homogeneous* in the following sense: if n -tuples \bar{a} and \bar{c} have the same types (i.e. the same values $\phi(\bar{a}) = \phi(\bar{c})$ for all L -formulas ϕ) then for every c_{n+1} and $\varepsilon > 0$ there is a tuple b_1, \dots, b_n, b_{n+1} of the same type with \bar{c}, c_{n+1} , so that $d(a_i, b_i) \leq \varepsilon$ for $i \leq n$. In fact for any n -tuples \bar{a} and \bar{b} there is an automorphism α of M such that

$$d(\alpha(\bar{c}), \bar{a}) \leq d(tp(\bar{a}), tp(\bar{c})) + \varepsilon.$$

(i.e M is *strongly ω -near-homogeneous* in the sense of Corollary 12.11 of [3]).

Definition 1.5 *A topological group G is called Roelcke precompact if for every open neighborhood of the identity U , there exists a finite subset $F \subset G$ such that $G = UFU$.*

The following theorem is a combination of the remark above, Theorem 6.2 of [18], Theorem 2.4 of [24] and Proposition 1.20 of [19].

Theorem 1.6 *Let G be the automorphism group of a non-compact separable structure M .*

Then

- (i) *the group G is approximately oligomorphic if and only if M is separably categorical;*
- (ii) *if G is Roelcke precompact and approximately oligomorphic for 1-orbits, then M is separably categorical;*
- (iii) *if the structure M is separably categorical, then G is Roelcke precompact.*

Axiomatizability in continuous logic, topological properties and sofic groups. Suppose \mathcal{C} is a class of metric L -structures. Let $Th^c(\mathcal{C})$ be the set of all closed L -conditions which hold in all structures of \mathcal{C} . It is proved in [3] (Proposition 5.14 and Remark 5.15) that every model of $Th^c(\mathcal{C})$ is elementary equivalent to some ultraproduct of structures from \mathcal{C} . Moreover by Proposition 5.15 of [3] we have the following statement.

The class \mathcal{C} is axiomatizable in continuous logic if and only if it is closed under metric isomorphisms and ultraproducts and its complement is closed under ultrapowers.

Let $Th_{\text{sup}}^c(\mathcal{C})$ be the set of all closed L -conditions of the form

$$\sup_{x_1} \sup_{x_2} \dots \sup_{x_n} \varphi = 0 \text{ (} \varphi \text{ does not contain } in f_{x_i}, \sup_{x_i} \text{)},$$

which hold in all structures of \mathcal{C} . Some standard arguments also give the following theorem.

Theorem 1.7 (1) *The class \mathcal{C} is axiomatizable in continuous logic if and only if it is closed under metric isomorphisms, ultraproducts and taking elementary submodels.*

(2) *The class \mathcal{C} is axiomatizable in continuous logic by $Th_{sup}^c(\mathcal{C})$ if and only if it is closed under metric isomorphisms, ultraproducts and taking substructures.*

It is worth noting that when one considers classes axiomatizable in continuous logic it is obviously assumed that all operations and predicates are uniformly continuous. This shows that some topological properties cannot be described (axiomatized) in continuous logic.

Some other obstacles arise from the fact that existential quantifiers cannot be expressed in continuous logic. For example consider the class of all metric groups which are discrete in their metrics (with id as continuity moduli). This class is not closed under metric ultraproducts but if we replace all metrics by the $\{0, 1\}$ -one we just obtain the (axiomatizable) class of all groups.

It may also happen that when we extend an axiomatizable class of structures with the $\{0, 1\}$ -metric¹ by (abstract) structures from this class with all possible (not only possible discrete) metrics we lose axiomatizability. A nice example of this situation is the class of non-abelian groups with $[0, 1]$ -metrics. For example there is a sequence of non-abelian groups $G_n \leq Sym(2^n + 3)$ with $G_n \cong \mathbb{Z}(2)^n \times S_3$ so that their metric ultraproduct with respect to Hamming metrics is abelian (an easy exercise).

Continuous axiomatizability appears in one of the most active areas in group theory as follows.

An abstract group is *sofic* if it is embeddable into a metric ultraproduct of finite symmetric groups with Hamming metrics.

Let \mathcal{S} be the class of complete id -continuous metric groups of diameter 1, which are embeddable as closed subgroups via isometric morphisms into a metric ultraproduct of finite symmetric groups with Hamming metrics. This class is axiomatizable by Theorem 1.7. We call it the class of *metric sofic groups*.

Corollary 1.8 *The class of metric sofic is sup-axiomatizable (i.e. by its theory Th_{sup}^c).*

It is folklore that any abstract sofic group can be embedded into a metric ultraproduct of finite symmetric groups as a discrete subgroup (see the proof of Theorem 3.5 of [16]). This means that the set of all abstract sofic groups consists of all discrete structures of the class \mathcal{S} .

¹in this case axiomatizability in continuous logic is equivalent to axiomatizability in first-order logic

2 Boundedness properties

It is worth noting that many classes from geometric group theory are just universal. For example if a group has free isometric actions on real trees (resp. Hilbert spaces) then any its subgroup has the same property. Similarly a closed subgroup of a locally compact amenable group is amenable.² Thus these classes are bountiful.

On the other hand if we extend these classes by non-compact locally compact groups without Kazhdan's property **(T)** or by groups admitting isometric actions on real trees without fixed points then we lose universality. Are these classes still bountiful? We may further extend our classes by so called *non-boundedness properties* introduced in [19]. For example consider metric groups which satisfy **non-OB** (in terms of [19]): they have isometric strongly continuous actions (i.e. the map $g \rightarrow g \cdot x$ defined on G is continuous for each x) on metric spaces with unbounded orbits. The first part of this section is devoted to some modifications of this property. We will show how continuous logic can work in these cases. In fact metric groups from these classes can be presented as reducts of continuous metric structures which induce some special actions.

In the second part of the section we consider **non-(T)** and **non-FR** (of fixed points for isometric actions on real trees). Note that **(T)** and property **FH** (that any strongly continuous isometric affine action on a real Hilbert space has a fixed point) are equivalent for σ -compact locally compact groups (see Chapter 2 in [1]). Since definitions of these properties require Hilbert spaces (or unbounded trees), we will here apply a many-sorted version of continuous logic (as in Section 15 of [3]). We will also present our groups as a union of an increasing chain of subsets of bounded diameters treating each subset as a sort. This situation is very natural if the group is σ -compact (i.e. a union of an increasing chain of compact subsets).

It is worth noting that by Section 1.10 of [19] in the case of σ -locally compact groups ($=\sigma$ -compact locally compact) Roelcke precompactness coincides with all boundedness properties studied in [19] excluding only **FH**. In particular it coincides with compactness and property **OB**. On the other hand an elementary submodel of a non-compact (resp. compact) continuous structure is also non-compact (resp. compact, see [3], Section 10). Thus by the Löwenheim-Skolem theorem the property **non-OB** is bountiful in the class of locally compact Polish groups (in a 1-sorted language). This explains why in the first part of the section we do not assume that a group is locally compact or Polish.

It is worth noting that our methods do not work for the classes of (locally compact) groups satisfying properties **(T)**, **FR** and **FH**. The case of amenable Polish groups also looks very interesting. A topological group G is

²the class of discrete initially amenable groups (see [8]) is universal too

called *amenable* if every G -flow admits an invariant Borel probability measure. In the case of locally compact groups this definition coincides with the classical one. It is noticed in [12], that the group $Sym(\omega)$ of all permutations of ω is amenable. Since it has closed non-amenable subgroups, the class of amenable Polish groups is not universal (with respect to taking closed subgroups).

2.1 Negations of strong boundedness and OB

An abstract group G is *Cayley bounded* if for every generating subset $U \subset G$ there exists $n \in \omega$ such that every element of G is a product of n elements of $U \cup U^{-1} \cup \{1\}$. If G is a Polish group then G is *topologically Cayley bounded* if for every analytic generating subset $U \subset G$ there exists $n \in \omega$ such that every element of G is a product of n elements of $U \cup U^{-1} \cup \{1\}$. It is proved in [18] that for Polish groups property **OB** is equivalent to topological Cayley boundedness together with *uncountable topological cofinality*: G is not the union of a chain of proper open subgroups.

Discrete groups. Let us consider the abstract (discrete) case. A group is *strongly bounded* if it is Cayley bounded and cannot be presented as the union of a strictly increasing chain $\{H_n : n \in \omega\}$ of proper subgroups (has *cofinality* $> \omega$). It is known that strongly bounded groups have property **FA**, i.e. any action on a simplicial tree fixes a point.

It is shown in [7], that strongly bounded groups have property **FH**. It can be also deduced from [7] that strongly bounded groups have property **FR** that every isometric action of G on a real tree has a fixed point (since such a group acting on a real tree has a bounded orbit, all the elements are elliptic and it remains to apply cofinality $> \omega$). It is now clear that the bountiful class of groups having free isometric actions on real trees (or on real Hilbert spaces) is disjoint from strong boundedness.

Proposition 2.1 *The following classes of groups are reducts of axiomatizable classes in $L_{\omega_1\omega}$:*

- (1) *The complement of the class of strongly bounded groups;*
- (2) *The class of groups of cofinality $\leq \omega$;*
- (3) *The class of groups which are not Cayley bounded;*
- (4) *The class of groups presented as non-trivial free products with amalgamation (or HNN-extensions);*
- (5) *The class of groups having homomorphisms onto \mathbb{Z} .*

*All these classes are bountiful. The class of groups which do not have property **FA** is bountiful too.*

Proof. (1) We use the following characterization of strongly bounded groups from [7].

A group is strongly bounded if and only if for every presentation of G as $G = \bigcup_{n \in \omega} X_n$ for an increasing sequence X_n , $n \in \omega$, with $\{1\} \cup X_n^{-1} \cup X_n \cdot X_n \subset X_{n+1}$ there is a number n such that $X_n = G$.

Let us consider the class \mathcal{K}_{nb} of all structures $\langle G, X_n \rangle_{n \in \omega}$ with the axioms stating that G is a group, $\{X_n\}$ is a sequence of unary predicates on G defining a strictly increasing sequence of subsets of G with $\{1\} \cup X_n^{-1} \cup X_n \cdot X_n \subset X_{n+1}$ (these axioms are first-order) and

$$(\forall x)(\bigvee_{n \in \omega} x \in X_n).$$

By the Löwenheim-Skolem theorem for countable fragments of $L_{\omega_1\omega}$ ([14], p.69) any subset C of such a structure is contained in an elementary submodel of cardinality $|C|$ (the countable fragment which we consider is the minimal fragment containing our axioms). This proves bountifulness in case (1).

(2) The case groups of cofinality $\leq \omega$ is similar.

(3) The class of groups which are not Cayley bounded is a class of reducts of all groups expanded by an unary predicate $\langle G, U \rangle$ with an $L_{\omega_1\omega}$ -axiom stating that U generates G and with a system of first-order axioms stating that there exists an element of G which is not a product of n elements of $U \cup U^{-1} \cup \{1\}$. The rest is clear.

(4) The class of groups which can be presented as non-trivial free products with amalgamation is the class of reducts of all groups expanded by two unary predicates $\langle G, U_1, U_2 \rangle$ with first-order axioms that U_1 and U_2 are subgroups and with $L_{\omega_1\omega}$ -axioms stating that $U_1 \cup U_2$ generates G and a word in the alphabet $U_1 \cup U_2$ is equal to 1 if and only if this word follows from the relators of the free product of U_1 and U_2 amalgamated over $U_1 \cap U_2$. The rest of (4) is clear.

(5) Groups having homomorphisms onto \mathbb{Z} can be considered as reducts of structures in the language $\langle \cdot, \dots, U_{-n}, \dots, U_0, \dots, U_m, \dots \rangle$, where predicates U_t denote preimages of the corresponding integer numbers.

To see that the class of groups without **FA** is bountiful, take any infinite $G \models \mathbf{notFA}$. It is well-known ([22], Section 6.1) that such a group belongs to the union of the classes from statements (2),(4) and (5). Thus G has an expansion as in one of the cases (2),(4) or (5). Now applying the Löwenheim-Skolem theorem, for any $C \subset G$ we find a subgroup of G of cardinality $|C|$ which contains C and does not satisfy **FA**. \square

Topological groups. As we already mentioned in Introduction separably categorical structures have Roelcke precompact automorphism groups. In the following definition we consider several versions of this property.

Definition 2.2 Let G be a topological group.

- (1) The group G is called *bounded* if for any open V containing 1 there is a finite set $F \subseteq G$ and a natural number $k > 0$ such that $G = FV^k$.
- (2) The group G is *Roelcke bounded* if for any open V containing 1 there is a finite set $F \subseteq G$ and a natural number $k > 0$ such that $G = V^k F V^k$.
- (3) The group G is *Roelcke precompact* if for any open V containing 1 there is a finite set $F \subseteq G$ such that $G = V F V$.
- (4) The group G has property $(\mathbf{OB})_k$ if for any open symmetric $V \neq \emptyset$ there is a finite set $F \subseteq G$ such that $G = (FV)^k$.

It is known that for Polish groups property \mathbf{OB} is equivalent to the property that for any open symmetric $V \neq \emptyset$ there is a finite set $F \subseteq G$ and a natural number k such that $G = (FV)^k$. Thus when G is non- \mathbf{OB} , there is a non-empty open V such that for any finite F and a natural number k , $G \neq (FV)^k$. Note that for such F and k there is a real number ε such that some $g \in G$ is ε -distant from $(FV)^k$. Indeed, otherwise $(FV)^k V$ would cover G .

This explains why in order to define a suitable class which is complementary to \mathbf{OB} we consider the following property.

Definition 2.3 A metric group G is called *uniformly non- \mathbf{OB}* if there is an open symmetric $V \neq \emptyset$ so that for any natural numbers m and k there is a real number ε such that for any m -element subset $F \subset G$ there is $g \in G$ which is ε -distant from $(FV)^k$.

Uniform non-boundedness, uniform non-Roelcke boundedness, uniform non-Roelcke precompactness and uniform non- $(\mathbf{OB})_k$ are defined by the same scheme.

It is clear that in the case discrete groups if a symmetric subset V has the property that $G \neq (FV)^k$ for all finite $F \subset G$ and natural numbers k , then the corresponding uniform version also holds.

Proposition 2.4 The following classes of metric groups are bountiful:

- (1) The class of uniformly non-bounded groups;
- (2) The class of uniformly non-Roelcke bounded groups;
- (3) The class of uniformly non-Roelcke precompact groups;
- (4) The class of uniformly non- $(\mathbf{OB})_k$ -groups;
- (5) The class of uniformly non- (\mathbf{OB}) -groups.

Proof. Let us consider the class of uniformly non- (\mathbf{OB}) -groups. Let \mathcal{K}_0 be the class of all continuous metric structures $\langle G, P, Q \rangle$ with the axioms stating that G is a group and $P : G \rightarrow [0, 1]$ and $Q : G \rightarrow [0, 1]$ are unary predicates on G with $Q(1) = 0$ so that

$$\sup_x \min(P(x), Q(x)) = \sup_x |P(x) - P(x^{-1})| = 0 \text{ and } \inf_x |P(x) - 1/2| = 0,$$

$$\sup_x |Q(x) - Q(x^{-1})| = 0 \text{ and } \inf_x |Q(x) - 1/2| = 0,$$

and for all rational $\varepsilon \in [0, 1]$

$$\sup_x \min(\varepsilon - Q(x), \inf_y (\max(d(x, y) - 2\varepsilon, \varepsilon - P(y))) = 0.$$

Note that the last axiom implies that any neighbourhood of an element from the nullset of Q contains an element with non-zero P .

For any natural m and k and any rational ε let us consider the following condition (say $\theta(m, k, \varepsilon)$):

$$\sup_{x_1 \dots x_m} \inf_x \sup_{y_1 \dots y_k} \min(P(y_1), \dots, P(y_n), (\varepsilon - \min_{w \in W_{m,k}} (d(x, w)))) = 0,$$

where $W_{m,k}$ consists of all words of the form $x_{i_1} y_1 x_{i_2} y_2 \dots x_{i_k} y_k$.

If G is a uniformly non-(OB)-group, then find an open symmetric V such that for any natural numbers m and k there is a real number ε such that for any m -element subset $F \subset G$ there is $g \in G$ which is ε -distant from $(FV)^k$. We interpret $Q(x)$ by $d(x, V)$ and $P(x)$ by $d(x, G \setminus V)$ (possibly normalizing them to satisfy the axioms of \mathcal{K}_0). Then observe that $\langle G, P, Q \rangle \in \mathcal{K}_0$ and for any natural numbers m and k there is a rational number ε so that $\theta(m, k, \varepsilon)$ holds in (G, P, Q) . By the Löwenheim-Skolem theorem for continuous logic any infinite subset C of such a structure is contained in an elementary submodel of the same density character as C . To verify uniform **non-(OB)** in such a submodel take the complement of the nullset of $P(x)$ as an open symmetric subset. This proves statement (5).

All remaining cases are considered in a similar way. \square

2.2 Unbounded actions

Negation of (T). Let a topological group G have a strongly continuous unitary representation on a Hilbert space \mathbf{H} . A closed subset $Q \subset G$ has an *almost ε -invariant unit vector* in \mathbf{H} if

$$\text{there exists } v \in \mathbf{H} \text{ such that } \sup_{x \in Q} \|x \circ v - v\| < \varepsilon \text{ and } \|v\| = 1.$$

We call a closed subset Q of the group G a *Kazhdan set* if there is ε with the following property: every unitary representation of G on a Hilbert space with almost (Q, ε) -invariant unit vectors also has a non-zero invariant vector. If the group G has a compact Kazhdan subset then it is said that G has property (T) of Kazhdan.

If we want to consider unitary representations in continuous logic we should fix continuity moduli for the corresponding binary functions $G \times B_n \rightarrow B_n$ induced by the action, where B_n is the n -ball of the corresponding Hilbert space. In fact if G is σ -locally compact, then we can present G as the union of a chain of compact subsets $K_1 \subseteq K_2 \subseteq \dots$ and consider continuity moduli for the corresponding functions $K_m \times B_n \rightarrow B_n$. Note that each B_k and K_l will be considered as sorts of a continuous structure. In this version of

continuous logic we do not assume that the diameter of a sort is bounded by 1. It can become any rational number.

We can now slightly modify the definition of a Kazhdan set as follows.

Definition 2.5 *Let G be the union of a chain of closed subsets $K_1 \subseteq K_2 \subseteq \dots$ of bounded diameters. Let $\mathcal{F} = \{F_1, F_2, \dots\}$ be a family of continuity moduli for continuous function $K_i \times B_i \rightarrow B_i$.*

We call a closed subset Q of the group G an \mathcal{F} -Kazhdan set if there is ε with the following property: every \mathcal{F} -continuous unitary representation of G on a Hilbert space with almost (Q, ε) -invariant unit vectors also has a non-zero invariant vector.

Let us consider such actions in continuous logic. We treat a Hilbert space over \mathbb{R} exactly as in Section 15 of [3]. We identify it with a many-sorted metric structure

$$(\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_r\}_{r \in \mathbb{R}}, +, -, \langle \rangle),$$

where B_n is the ball of elements of norm $\leq n$, $I_{mn} : B_m \rightarrow B_n$ is the inclusion map, $\lambda_r : B_m \rightarrow B_{km}$ is scalar multiplication by r , with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |r| < k$; furthermore, $+, - : B_n \times B_n \rightarrow B_{2n}$ are vector addition and subtraction and $\langle \rangle : B_n \rightarrow [-n^2, n^2]$ is the inner product. The metric on each sort is given by $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$. For every operation the continuity modulus is standard.

Stating existence of infinite approximations of orthonormal bases (by a countable family of axioms, see Section 15 of [3]) we assume that our Hilbert spaces are infinite dimensional. By [3] they form the class of models of a complete theory which is κ -categorical for all infinite κ , and admits elimination of quantifiers.

This approach can be naturally extended to complex metric spaces. We only have to extend the family $\lambda_r : B_m \rightarrow B_{km}$, $r \in \mathbb{R}$, to a family $\lambda_c : B_m \rightarrow B_{km}$, $c \in \mathbb{C}$, of scalar multiplications by $c \in \mathbb{C}$, with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |c| < k$.

Let us consider a class of continuous metric structures which are unions of the many-sorted structures

$$(\{K_n\}_{n \in \omega}, \cdot, {}^{-1}, 1),$$

corresponding to groups G presented as $\bigcup_{n \in \omega} K_n$, together with metric structures of complex Hilbert spaces

$$(\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{C}}, +, -, \langle \rangle).$$

The operation \cdot (and ${}^{-1}$) is considered as a family of maps $K_n \times K_n \rightarrow K_{n+1}$. (maps $K_n \rightarrow K_{n+1}$ respectively), $n \in \omega$. Such a structure (say $A(G, \mathbf{H})$) also

contains a binary operation \circ of an action which is defined by a family of appropriate maps $K_n \times B_m \rightarrow B_m$. When we add the obvious continuous *sup*-axioms that the action is linear and unitary, we obtain an axiomatizable class \mathcal{K}_{GH} . We do not state exactly which continuity moduli would correspond to these operations. In fact this depends on groups and actions we want to have in \mathcal{K}_{GH} . By $\mathcal{K}_{GH}(\mathcal{F})$ we denote the corresponding class with continuity moduli \mathcal{F} .

Assuming that continuity moduli \mathcal{F} are fixed let $\mathcal{K}_{aiv}(\mathcal{F})$ be the subclass of \mathcal{K}_{GH} axiomatizable by the axioms

$$\inf_{v \in B_m} \sup_{x \in K_n} \max(\|x \circ v - v\| - \frac{1}{n}, |1 - \|v\||) = 0, \quad m, n \in \omega \setminus \{0\},$$

which in fact say that each K_n has an almost $\frac{1}{n}$ -invariant unit vector in \mathbf{H} .

Below we will only consider metric groups which have presentations $G = \bigcup_{i \in \omega} K_i$, where $\{K_i : i \in \omega\}$ is an increasing sequence of closed subsets of diameters $d_1 \leq d_2 \leq \dots$ with $\{1\} \cup K_n \cdot K_n \cup K_n^{-1} \subseteq K_{n+1}$. Moreover we will assume that every function $K_n \times K_n \rightarrow K_{n+1}$ induced by the multiplication is uniformly continuous with respect to some fixed family of continuity moduli \mathcal{F}_0 . When G is a σ -locally compact group then such \mathcal{F}_0 and such a decomposition obviously exist.

Proposition 2.6 *Let G be a metric group having a presentation $G = \bigcup_{i \in \omega} K_i$ into an increasing sequence of closed subsets of bounded diameters as above, so that no K_i is an \mathcal{F} -Kazhdan set for G . Then there is a metric structure in $\mathcal{K}_{aiv}(\mathcal{F})$ which naturally expands $(\{K_n\}_{n \in \omega}, \cdot, {}^{-1}, 1)$ so that G does not have non-zero fixed vectors. In particular any σ -locally compact group G without property **(T)** for \mathcal{F} -actions has such an expansion to \mathbf{H} .*

Proof. Consider the particular case of the proposition. Let $G = \bigcup_{i \in \omega} K_i$, where $\{K_i : i \in \omega\}$ is an increasing sequence of compact neighborhoods of 1 with $K_n \cdot K_n \cup K_n^{-1} \subseteq K_{n+1}$. We know that for any natural n and rational $0 < q < 1$ there is a unitary \mathcal{F} -continuous representation of G on a Hilbert space \mathbf{H} which has an almost (K_n, q) -invariant unit vector but does not have a non-zero invariant vector. Decomposing a basis of \mathbf{H} into a union of an infinite family of pairwise disjoint infinite subsets (labelled by pairs (n, q)) and defining representations above on the corresponding subspaces, we find an unitary \mathcal{F} -representation of G on \mathbf{H} without non-zero invariant vectors so that for any natural n and rational $q < 1$ there is an almost (K_n, q) -invariant unit vector in \mathbf{H} .

Let us define a required metric structure (denoted by $A(G, \mathbf{H})$ as above) as a union of the many-sorted structure

$$(\{K_n\}_{n \in \omega}, \cdot, {}^{-1}, 1),$$

corresponding to the group G , together with the metric structure of the separable Hilbert space

$$(\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{C}}, +, -, \langle \rangle).$$

The operation \cdot (and $^{-1}$) is considered as a family of maps $K_n \times K_n \rightarrow K_{n+1}$. (maps $K_n \rightarrow K_{n+1}$ respectively), $n \in \omega$. The structure $A(G, \mathbf{H})$ also contains a binary operation \circ of an action which is defined by a family of appropriate maps $K_n \times B_m \rightarrow B_m$. Since the representation is unitary, any element of G preserves each B_m . The axioms of $\mathcal{K}_{inv}(\mathcal{F})$ obviously hold in $A(G, \mathbf{H})$.

The argument in the general case is the same. \square

Remark 2.7 When any compact subset of G is contained in some K_i , the group G from the formulation above does not have property **(T)** for \mathcal{F} -continuous representations. For example this happens when each K_i is the closed d_i -ball of 1.

The following corollary can be considered as a kind of bountifulness for groups without **(T)**. It follows from the proposition and remark above and the Löwenheim-Skolem theorem for continuous logic, when $|L| = 2^{\aleph_0}$.

Corollary 2.8 *Let G be a metric group having a presentation $G = \bigcup_{i \in \omega} K_i$ into an increasing sequence of closed balls of 1 of bounded diameters, so that no K_i is an \mathcal{F} -Kazhdan set for G . Then for any subset $C \subset G$ with $\text{density}(C) \geq 2^{\aleph_0}$ there is a closed elementary (in continuous logic) subgroup of G containing C , with the same density character as C and without property **(T)** for \mathcal{F} -continuous representations.*

Non-FH-actions. To consider non-FH let us fix a binary function $\nu : \omega \times \omega \rightarrow \omega$ which is increasing in each argument. We now define $\mathcal{K}_{GH}(\nu)$, a class of continuous metric structures which are unions of many-sorted \mathcal{F}_0 -continuous structures

$$(\{K_n\}_{n \in \omega}, \cdot, ^{-1}, 1),$$

corresponding to groups G presented as $\bigcup_{n \in \omega} K_n$ (with assumptions as before Proposition 2.6), together with metric structures of real Hilbert spaces

$$(\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{R}}, +, -, \langle \rangle).$$

We also add a binary \mathcal{F} -continuous operation \circ of an action defined by a family of appropriate maps $K_n \times B_m \rightarrow B_{\nu(n,m)}$. Obvious continuous *sup*-axioms that the action is isometric give an axiomatizable class $\mathcal{K}_{GH}(\nu, \mathcal{F})$.

The following statement is a straightforward application of the continuous Löwenheim-Skolem theorem.

Proposition 2.9 *Let G be a metric group having a presentation $G = \bigcup_{i \in \omega} K_i$ into an increasing sequence of closed subsets of bounded diameters, so that there is an isometric action of G on a real Hilbert space inducing a structure from $\mathcal{K}_{GH}(\nu, \mathcal{F})$ without fixed points. Then for any subset $C \subset G$ with $\text{density}(C) \geq 2^{\aleph_0}$ there is a closed continuously elementary subgroup of G containing C , with the same density character as C and without property **FH** for \mathcal{F} -actions.*

Note that any σ -locally compact metric group G without property **FH** for \mathcal{F} -actions belongs to the class $\mathcal{K}_{GH}(\nu, \mathcal{F})$ (for appropriate ν and \mathcal{F}_0). We will now see this in a slightly stronger form. Fix a function η which assigns to a natural number k a pair $(l, s) \in \omega \times \omega$. To obtain $\mathcal{K}_{nFH}(\nu, \eta, \mathcal{F})$ take the subclass of $\mathcal{K}_{GH}(\nu)$ axiomatizable by the axioms

$$\sup_{v \in B_k} \inf_{x \in K_l} \left(\frac{1}{s} \cdot \|x \circ v - v\| \right) = 0, \text{ for } \eta(k) = (l, s)$$

(saying that each vector of B_k is moved by some element of K_l by approximately $\frac{1}{s}$).

Proposition 2.10 *For any σ -locally compact metric group G without property **FH** for \mathcal{F} -continuous actions there are functions ν and η so that G belongs to the class $\mathcal{K}_{nFH}(\nu, \eta, \mathcal{F})$.*

Proof. Let $G = \bigcup_{i \in \omega} K_i$, where $\{K_i : i \in \omega\}$ is an increasing sequence of compact neighborhoods of 1 with $K_n \cdot K_n \cup K_n^{-1} \subseteq K_{n+1}$. Fix appropriate \mathcal{F}_0 . We know that there is an affine \mathcal{F} -continuous isometric action of G on a real Hilbert space **H** which does not have a fixed vector. Since for every $v \in \mathbf{H}$ the map

$$G \rightarrow \mathbf{H}, g \rightarrow gv$$

is continuous there are $k \in \omega$ and an open subset of G which maps 0 into B_k . In particular for any n we can find such a k so that all elements of K_n map 0 into B_k . This means that $K_n \circ B_m \subseteq B_{k+m}$. This defines a function $\nu : \omega \times \omega \rightarrow \omega$ so that the continuous structure corresponding to the action belongs to $\mathcal{K}_{GH}(\nu, \mathcal{F})$.

Let us show that this structure belongs to $\mathcal{K}_{nFH}(\nu, \eta, \mathcal{F})$ for appropriate η . Since G does not fix any point, each orbit of G is unbounded (Proposition 2.2.9 of [1]). Thus there is $g \in G$ so that $B_k \cap g(B_k) = \emptyset$. In particular there is $s \in \mathbb{N}$ such that $\frac{1}{s} \leq \|g \circ v - v\|$ for all $v \in B_k$. We define $\eta(k)$ to be (l, s) , where l is chosen so that K_l contains g as above. \square

Non-FR-actions. To consider non-FR we apply similar ideas. Let us fix a binary function $\nu : \omega \times \omega \rightarrow \omega$ which is increasing in each argument. We now

define $\mathcal{K}_{GR}(\nu, \mathcal{F})$, a class of continuous metric structures which are unions of the many-sorted structures

$$(\{K_n\}_{n \in \omega}, \cdot, ^{-1}, 1),$$

corresponding to groups G presented as $\bigcup_{n \in \omega} K_n$, together with metric structures of pointed real trees

$$(\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}),$$

where B_n is the n -ball of 0 in the tree. It is shown in [6] that the class of pointed real trees is axiomatizable in continuous logic by axioms of 0-hyperbolicity and the approximate midpoint property.

We also add a binary operation \circ of an action defined by a family of appropriate \mathcal{F} -continuous maps $K_n \times B_m \rightarrow B_{\nu(n,m)}$. Obvious continuous *sup*-axioms that the action is isometric give an axiomatizable class $\mathcal{K}_{GR}(\nu, \mathcal{F})$.

As in the non-**FH**-case we have the following straightforward statement.

Proposition 2.11 *Let G be an \mathcal{F}_0 -continuous group with respect to a presentation $G = \bigcup_{i \in \omega} K_i$ into an increasing sequence of closed subsets of bounded diameters, so that there is an isometric action of G on a real tree inducing a structure from $\mathcal{K}_{GR}(\nu, \mathcal{F})$ without fixed points. Then for any infinite subset $C \subset G$ there is a closed continuously elementary subgroup of G containing C , with the same density character as C and without property **FR** for \mathcal{F} -continuous actions.*

Note that since the language is countable we do not assume in Proposition 2.11 that $\text{density}(C) \geq 2^{\aleph_0}$.

Fix a function η which assigns to a natural number k a pair $(l, s) \in \omega \times \omega$. To obtain $\mathcal{K}_{nFR}(\nu, \eta, \mathcal{F})$ take the subclass of $\mathcal{K}_{GR}(\nu)$ axiomatised by the axioms

$$\sup_{v \in B_k} \inf_{x \in K_l} \left(\frac{1}{s} \cdot d(x \circ v, v) \right) = 0, \text{ for } \eta(k) = (l, s)$$

(saying that each element of B_k is moved by some element of K_l by approximately $\frac{1}{s}$).

Proposition 2.12 *For any σ -locally compact metric group G without property **FR** for strongly \mathcal{F} -continuous actions there are functions ν and η so that G belongs to the class $\mathcal{K}_{nFR}(\nu, \eta, \mathcal{F})$.*

Proof. Let $G = \bigcup_{i \in \omega} K_i$, where $\{K_i : i \in \omega\}$ is an increasing sequence of compact neighborhoods of 1 with $K_n \cdot K_n \cup K_n^{-1} \subseteq K_{n+1}$. Find appropriate moduli \mathcal{F}_0 making G an \mathcal{F}_0 -continuous structure. We know that there is an \mathcal{F} -continuous isometric action of G on a real tree T which does not have a fixed point. Since for every $v \in T$ the map

$$G \rightarrow T, g \rightarrow gx$$

is continuous there are $k \in \omega$ and an open subset of G which maps 0 into B_k . In particular for any n we can find a k so that all elements of K_n map 0 into B_k . This means that $K_n \circ B_m \subseteq B_{k+m}$. This defines a function $\nu : \omega \times \omega \rightarrow \omega$ so that the continuous structure corresponding to the action belongs to $\mathcal{K}_{GR}(\nu, \mathcal{F})$.

Let us show that this structure belongs to $\mathcal{K}_{nFR}(\nu, \eta, \mathcal{F})$ for appropriate η . If G has a hyperbolic element g of hyperbolic length r (i.e. there is a line L so that g r -shifts all points of L), then η is constant where l is chosen so that K_l contains this hyperbolic element and s is chosen so that $\frac{1}{s} \leq r$.

Consider the case when G consists of elliptic elements (i.e. fixing points). Since G does not fix any point, by a well-known argument G fixes an end ([22], Section 6.5, Exercise 2). Let L_0 be the half-line starting from 0 which represents this end and let v_1, \dots, v_i, \dots be a cofinal ω -sequence in L_0 with $d(v_i, v_{i+1}) \geq 1$. Then we may assume that G is the union of a strictly increasing chain of stabilizers G_i of v_i . Since the action is continuous, all G_i are closed.

Having k find j with $v_{j-1} \notin B_k$ (thus $v_j \notin B_k$). Since any arc linking v_j with an element from B_k must contain v_{j-1} , we see that if $g \in G_j$ fixes a point of B_k then it fixes v_{j-1} . In particular G_j does not fix any element of B_k . Since $d(v_j, v_{j-1}) \geq 1$ any point of B_k can be taken by some element of G_j at a distance greater than 1. Thus to define $\eta(k) = (l, s)$, we choose l so that K_l contains an element of G_j not fixing v_{j-1} . We define $s = 1$. \square

3 Separably categorical locally compact groups

If G is a locally compact group, then G admits a compatible complete left invariant metric $d(x, y)$ ([2], 3.C.2). We may assume that $d(x, y)$ satisfies $d(x, y) < 1$ (it can be replaced by $\frac{d(x, y)}{d(x, y) + 1}$). We thus may consider locally compact groups as a class of continuous metric structures

$$(G, d, \cdot, ^{-1}, 1),$$

together with fixed continuity moduli for functional symbols.

Theorem 3.1 *Let G be a separably categorical locally compact non-compact group. Then there is a compact clopen subgroup $H < G$ which is invariant with respect to all metric automorphisms of G , and the induced action of $\text{Aut}(G, d)$ on the coset space G/H is oligomorphic.*

If the connected component of the unity G^0 is not trivial, H can be taken to be G^0 . In this case and in the case when d is two-sided-invariant, the subgroup H is normal and G/H is an ω -categorical discrete group.

We start with the following preliminaries. We may assume that G is not discrete. There is a rational number $\rho < 1$ such that the ρ -ball of the unity

$B_\rho(1) = \{x \in G : d(x, 1) \leq \rho\}$ is compact. In particular $B_\rho(1)$ is a subset of $\text{acl}(\emptyset)$ in G (the condition $d(x, 1) \leq \rho$ defines a totally bounded, complete subset in any elementary extension of G). Thus any $B_\rho^n(1)$ also is a subset of $\text{acl}(\emptyset)$. Let G_ρ be the subgroup generated by $B_\rho(1)$. Note that for any $g \in G_\rho$ the open ball

$$B_{<\rho}(g) = \{x \in G : d(x, g) < \rho\} = \{x \in G : d(g^{-1}x, 1) < \rho\}$$

is a subset of G_ρ ; thus G_ρ is an open (in fact clopen) subgroup. If G has a non-trivial connected component of the unity G^0 , we may assume that $G_\rho = G^0$. Note that when d is a two-sided-invariant metric, G_ρ is a normal subgroup of G .

Lemma 3.2 *Assume that G is a separably categorical locally compact group. Then under the circumstances above the predicate $P(x) = d(x, G_\rho)$ is definable in G .*

Proof. Since the space $(S_n(T), d)$ is compact, for every ε there is a finite set of types which is ε -dense in the set of types of elements of $\bigcup_{n>0} B_\rho^n(1)$. Thus there is a number n such that the ε -neighbourhood of $B_\rho^n(1)$ contains the zeroset of $P(x) = \text{dist}(x, G_\rho)$. If (N, Q) is an elementary extension of (G, P) then (N, Q) satisfies the condition

$$\sup_x \inf_{y_1 \dots y_n} \max(d(y_1, 1) \dot{-} \rho, \dots, d(y_n, 1) \dot{-} \rho, |Q(x) - d(x, y_1 \dots y_n)| \dot{-} \varepsilon) = 0,$$

i.e. the ε -neighbourhood of $B_\rho^n(1)$ contains the zeroset of $Q(x)$. In particular the zeroset of Q coincides with the closure of G_ρ , i.e. is G_ρ itself and is a subset of $\text{acl}(\emptyset)$. Since (N, Q) is an elementary extension of (G, P) , $Q(x)$ is the distance from the zeroset of Q (see Theorem 9.12 in [3]). In particular any automorphism of N preserves Q . Using Corollary 9.11 of [3] (cited in Introduction above) we see that $P(x)$ is a definable predicate. \square

Lemma 3.3 *Under the circumstances above there is a natural number n so that $G_\rho = B_\rho^n(1)$. In particular G_ρ is compact.*

Proof. If $G_\rho \neq B_\rho^n(1)$ for all $n \in \omega$, there are positive rational numbers $\varepsilon_1, \dots, \varepsilon_n, \dots$ so that the ε_n -neighbourhood of $B_\rho^n(1)$ does not cover G_ρ . Thus all statements

$$\sup_{x_1 \dots x_n} (\min(\varepsilon_n \dot{-} d(x, x_1 \cdot \dots \cdot x_n), \rho \dot{-} d(1, x_1), \dots, \rho \dot{-} d(1, x_n))) = 0$$

are finitely consistent together with $P(x) = 0$. By compactness of continuous logic we obtain a contradiction. \square

Since G_ρ is a characteristic subgroup of G with respect to the automorphism group of the metric structure G , we see that $\text{Aut}(G, d)$ acts correctly on G/G_ρ by permutations of G/G_ρ . Note that G/G_ρ is a discrete space with respect to the topology induced by the topology of G .

Lemma 3.4 *The action of $\text{Aut}(G, d)$ on G/G_ρ is oligomorphic.*

Proof. Since (G, d) is separably categorical, $\text{Aut}(G, d)$ is approximately oligomorphic on (G, d) . Thus for every n there is a finite set F of n -tuples from G such that the set of orbits meeting F is ρ -dense in (G, d) . In particular for any $g_1, \dots, g_n \in G$ there is a tuple $(h_1, \dots, h_n) \in F$ and an automorphism $\alpha \in \text{Aut}(G, d)$ such that $g_i^{-1}\alpha(h_i) \in G_\rho$ for all $i \leq n$. \square

To see that Theorem 3.1 follows from lemmas above just take H to be G_ρ .

References

- [1] B.Bekka, P.de la Harpe and A.Valette, *Kazhdan's Property (T)*, New Mathematical Monographs, 11, Cambridge University Press, Cambridge, 2008
- [2] H.Becker, *Polish Group Actions: dichotomies and generalized elementary embeddings*, J.Amer. Math. Soc. 11(1998), 397 - 449.
- [3] I.Ben Yaacov, A.Berenstein, W.Henson and A.Usvyatsov, *Model theory for metric structures*. In: Model theory with Applications to Algebra and Analysis, v.2 (Z.Chatzidakis, H.D.Macpherson, A.Pillay and A.Wilkie, eds.), London Math. Soc. Lecture Notes, v.350, pp. 315 - 427, Cambridge University Press, 2008.
- [4] G.Bergman, *Generating infinite symmetric groups*, Bull.London Math.Soc. **38**(2006), 429 - 440.
- [5] P.Cameron, *Oligomorphic Permutation Groups*, London Math. Soc. Lecture Notes, vol. 152, Cambridge University Press, 1990.
- [6] S.Carlsle, *Model theory of real trees and their isometries*, PhD thesis, University of Illinois at Urbana-Champaign, 2009 (available at <http://www.math.uiuc.edu/~henson/cfo/carlslethesis.pdf>).
- [7] Y.de Cornulier, *Strongly bounded groups and infinite powers of finite groups*, Comm. Algebra, **34**(2006), 2337 - 2345.
- [8] Y.de Cornulier, *A sofic group away from amenable groups*, Math. Ann., 350(2011), 269 - 275.
- [9] G.Elek and E.Szabo, *Hyperlinearity, essentially free actions and L^2 -invariants. The sofic property*, Math. Ann., 332(2005), 421 - 441.
- [10] P.de la Harpe and A.Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts*, Astérisque **175**, SMF, 1989.

- [11] S.Hernandes, K.H.Hofmann and S.A.Morris, *The weights of closed subgroups of a locally compact subgroup*, arXiv: 1201.3814.
- [12] A.Kechris, *Dynamic of non-archimedean Polish groups*, to appear in Proceedings of the European Congress in Mathematics, Krakow, Poland, 2012 (available at <http://www.math.caltech.edu/papers/nonarchimedean02.pdf>)
- [13] A.Kechris and Ch.Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. London Math. Soc.(3), 94(2007), 302 - 350.
- [14] J.H.Keisler, *Model Theory for Infinitary Logic*, North-Holland, London, 1971.
- [15] R.D.Kopperman and A.R.D.Mathias, *Some problems in group theory*, In: The syntax and semantics of infinitary languages, Lect. Notes in Math., 72(1968), 131 - 138.
- [16] V.Pestov, *Hyperlinear and sofic groups: a brief guide*, Bull. Symb. Logic, 14(2008), 449 - 480.
- [17] R.E.Phillips, *Countably recognizable classes of groups*, Rocky Mountain J. Math., 1(1971), 489 - 497.
- [18] Ch.Rosendal, *A topological version of the Bergman property*, Forum Math. 21(2009), 299 - 332.
- [19] Ch.Rosendal, *Global and local boundedness of Polish groups*, arXiv: 1203.604.
- [20] G.Sabbagh, *On properties of countable character*, Bull. Austr. Math. Soc., 4(1971), 183 - 192.
- [21] K.Schoretsanitis, *Fraïssé Theory for Metric Structures*, PhD thesis, University of Illinois at Urbana-Champaign, 2007 (available at <http://www.math.uiuc.edu/~henson/cfo/metricfraise.pdf>).
- [22] J. Serre, *Trees*, Springer, NY, 1980.
- [23] S.Thomas, *Complete existentially closed locally finite groups*, Arch. Math. 44(1985), 97 - 109.
- [24] T.Tsankov, *Unitary representations of oligomorphic groups*, arXiv: 1101.219.

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